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Inverse structural modification using constraints

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Abstract

In a structural modification problem the mass and stiffness matrices are modified to obtain a desired spectrum. In this paper, this is done by imposing constraints on the structure. The undamped natural vibrations of a constrained linear structure are calculated by solving a generalized eigenvalue problem derived from the equations of motion for the constrained system involving Lagrangian multipliers. The coefficients of the constraint matrix are taken as design variables and a set of equations defining the inverse structural modification problem is formulated. This modification problem requires an iterative method for its solution. An algorithm based on Newton's method is employed. Each iteration step involves the calculation of a rectangular Jacobian and the solving of an associated underdetermined system of linear equations. The system can be solved by using the Moore–Penrose inverse. The method is demonstrated in some numerical examples.

1. Introduction

Today, with the evolution of computer hardware and matrix computation procedures, it is routine to predict the dynamic behaviour of structures using numerical methods, and this has replaced many of the costly experimental modal analyses. If an analysis reveals that the structure has unwanted modal properties there is a need to modify the structure in order to obtain the desired modal properties. One procedure, where changes in mass- stiffness- or damping-properties aim at a change in the performance of the structural resonances, is called a structural modification.

Theoretically, structural modifications can be performed in a variety of ways depending on the type of structure considered. Early structural modification techniques included substructuring techniques; see for instance Refs. [1–3], where the structure was divided into substructures, resulting in smaller systems that were modified individually. These were then coupled together into the global structure and the technique was very appropriate for structures consisting of repeating substructures. Other ways of modifying a structure could be to specify the modification matrices and then update the modal properties using truncated modal mass and stiffness matrices, resulting in Rayleigh–Ritz approximations, i.e. upper bounds of the modified spectra are obtained. However, these upper bounds may not always be good approximations of the actual spectra. In fact, they can be very misleading. A lengthy discussion of this problem can be found in Ref. [4]. This approach was,

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nevertheless, used by Ram and Braun [5] who formulated expressions, which gave upper and lower bounds of the lower part of the spectrum. An alternative approach is to impose constraints on the structure. This can be done as demonstrated by Kerstens [6,7], Dowell [8] and Lidström and Olsson [9].

The afore-mentioned approaches can all be classed as *direct modification approaches*, i.e. the modifications are imposed on the structure and the new modal properties are obtained by calculations and measurements. This procedure is repeated until the requirements are met. The opposite of the direct modification approach is the *inverse modification approach* where the desired modal properties are specified and the mass and stiffness matrices are modified so that the modal requirements are met. Braun and Ram [10] analyzed structures consisting of discrete springs and masses and derived an approximate method for calculating the modification matrices of the structure. The approximation was, similar to that mentioned above; optimal in a Rayleigh–Ritz sense, but it was not necessarily a good approximation. Due to the fact that the number of design variables is, in general, higher than, or at least equal to, the number of specified desired eigenvalues the solution is not unique. Hence, there exist numerous possible solutions, not all physically realizable. Gladwell and Zhu [11] and Gladwell [12–14] studied finite element and spring and mass structures with tridiagonal mass and stiffness matrices and derived a closed form procedure, cf. Ref. [14], to construct a structure with minimal mass satisfying the desired spectrum.

For many structures, when using, for instance, the finite element method, the mass and the stiffness matrices are coupled, i.e. both matrices will be affected by a modification of the design variables. The problem then becomes more complicated and, in general, the different types of elements and design variables considered cannot be dealt with in a general manner, and must therefore be regarded individually. Djoudi et al. [15] derived a simple eigenvalue problem for calculating the inverse modification problem for bar and truss structures. This method is free from iterations and determines the cross sectional area modifications of the members involved. However, the solutions of the eigenvalue problem are not necessarily physically realizable, i.e. the resulting cross sectional area can be negative. The method may also lead to an unwanted spectral shift. A similar formulation was derived by Bahai et al. [16] and Bahai and Aryana [17] for continuous finite elements with design variables such as Young's modulus, thicknesses and nodal coordinates. However, the solution scheme included a first-order Taylor expansion, and hence the solution obtained is an approximation. In Refs. [18,19], Farahani and Bahai provided algorithms for relocating the spectrum for arbitrary finite element structures. The algorithms were based on first (cf. Ref. [18]) and second (cf. Ref. [19]) order expansions of the eigenvalues with respect to the design variables, and the numerical examples demonstrated high accuracy. In Ref. [20], Kim et al. gave an algorithm for the inverse eigenvalue problem, which is based on an initial approximation that is obtained by linear perturbation of the system. The solution is then improved through iterations with higher order perturbation theory until convergence is met, resulting in highly accurate solutions.

Joseph [21] formulated an inverse generalized eigenvalue problem for arbitrary coupled mass and stiffness matrices. These were assumed to be coupled via design variables in an arbitrary manner. Thus the system of equations could in practice become nonlinear and hence the solutions were obtained in an iterative manner by employing Newton's method. In order to use Newton's method, the Jacobian had to be calculated which was easily done for the symmetric generalized eigenvalue problem, cf. Ref. [22]. Furthermore, the method was applied to linear truss structures using the cross sectional area of the members as design variables and it displayed good convergence. Formulations similar to Ref. [21] can be found in Refs. [23,24] with new solution schemes.

Due to the fact that the Jacobian, in general, is not quadratic, the solution of the linear system of equations, determining the next iterate, is not unique nor does it necessarily satisfy any lower bounds of the design variables. This problem was addressed by Sivan and Ram [25] who delivered an algorithm which solves the nonlinear underdetermined system of equations, resulting in design variables that satisfy certain lower bounds.

In this paper, we will impose linear constraints on the structure in an inverse manner in order to obtain a desired spectrum. The nonsymmetric constraint formulation given in Ref. [9] will then be used with the constraint matrix elements as design variables, analogous to what has been done in Ref. [21]. We then formulate the Jacobian, which is derived and proven in a theorem in order to solve the problem iteratively. The method is illustrated in a number of simple numerical examples where an algorithm similar to the one provided in Ref. [25] is used.

The outline of the paper is as follows: in the following section a brief summary of the mathematical notation used in this paper is given. Thereafter we give a brief discussion covering some basics of modal analysis. In Section 4 the basics of the constraint formulation in Ref. [9] is briefly summarized. This is followed by the inverse eigenvalue formulation and the derivation of the Jacobian. Thereafter, the algorithm and the solution of the underdetermined system are discussed, and finally we give a few simple numerical examples and a summary of the paper.

2. Notation

In this paper, $\mathbb{R}$ denotes the set of real numbers. The set of n-dimensional, real column vectors is denoted by $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$ and the null vector is written $\mathbf{0}_n$. The Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is denoted $\|\mathbf{x}\| = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$. $\mathbb{R}^{m\times n}$ denotes the set of real matrices of order $m \times n$ with the null matrix written $\mathbf{0}_{m\times n}$. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ then $\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ is the transpose of $\mathbf{A}$. The rank of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is written rank($\mathbf{A}$). If $\mathbf{A}$ is a square matrix, i.e., $n = m$, then det($\mathbf{A}$) denotes the determinant of $\mathbf{A}$ and if det($\mathbf{A}$)$\neq$0 then $\mathbf{A}^{-1}$ denotes its inverse. $\mathbf{I}_{n\times n}$ denotes the identity matrix in $\mathbb{R}^{n\times n}$. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, then the following linear spaces associated with $\mathbf{A}$ will be employed:

$$\mathrm{range}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^m | \mathbf{x} = \mathbf{A}\mathbf{u},\ \mathbf{u} \in \mathbb{R}^n\}, \quad \mathrm{kernel}(\mathbf{A}) = \{\mathbf{u} \in \mathbb{R}^n | \mathbf{A}\mathbf{u} = \mathbf{0}_m\}.$$

If V is a linear subspace of $\mathbb{R}^n$ then the dimension of V is denoted dim(V) and the orthogonal complement of V, $V^{\perp}$, is defined by

$$V^{\perp} = \left\{\mathbf{y} \in \mathbb{R}^n \middle| \mathbf{y}^{\mathrm{T}}\mathbf{x} = 0, \quad \forall \mathbf{x} \in V\right\}.$$

3. Preliminaries

Consider the free, undamped vibrations of an n degree-of-freedom (dof) mechanical structure. This is modelled by the system of linear, second-order differential equations

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}_n, \tag{1}$$

with a mass matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ which, throughout this paper, is assumed to be symmetric and positive definite, and a stiffness matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$ which is assumed to be symmetric and positive semi-definite. The configuration coordinates of the structure are given by the vector $\mathbf{q} = [q_1 q_2 \ldots q_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\mathbf{q} = \mathbf{q}(t)$. A solution to Eq. (1) is given by

$$\mathbf{q} = \mathbf{x} \sin \omega t, \tag{2}$$

where the constant amplitude vector $\mathbf{x} \in \mathbb{R}^n$ and the angular frequency ω satisfy the linear system of equations defining the generalized eigenvalue problem

$$(-\omega^2\mathbf{M} + \mathbf{K})\mathbf{x} = \mathbf{0}_n, \tag{3}$$

where $\mathbf{x}$ is referred to as the mode shape.

The existence of nontrivial solutions, $\mathbf{x}\neq\mathbf{0}$, to Eq. (3) requires that ω satisfies the secular equation:

$$\det(-\omega^2\mathbf{M} + \mathbf{K}) = 0, \tag{4}$$

where the roots of Eq. (4), the natural frequencies of the structure, and the corresponding mode shapes are denoted by

$$0 \leqslant \omega_1^2 \leqslant \omega_2^2 \leqslant \cdots \leqslant \omega_n^2 \quad \text{and} \quad \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \tag{5}$$

respectively, so that

$$(-\omega_i^2\mathbf{M} + \mathbf{K})\mathbf{x}_i = \mathbf{0}_n, \quad i = 1, 2, \ldots, n \tag{6}$$

is satisfied. The mode shapes $\mathbf{x}_1$, $\mathbf{x}_2$,…,$\mathbf{x}_n$ may be chosen as linearly independent, satisfying $\mathbf{x}_i^{\mathrm{T}}\mathbf{M}\mathbf{x}_j = 0$, $i\neq j$. It is convenient to assemble the mode shapes in the nonsingular modal matrix $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_n]$, and the corresponding natural frequencies in the diagonal spectral matrix $\mathbf{\Omega}^2 = \mathrm{diag}(\omega_1^2 \ \ \omega_2^2 \ldots \omega_n^2)$. A normalization of the modal matrix with respect to the mass matrix is obtained using the requirement $\mathbf{X}^{\mathrm{T}}\mathbf{M}\mathbf{X} = \mathbf{I}_{n\times n}$, and then $\mathbf{X}^{\mathrm{T}}\mathbf{K}\mathbf{X} = \mathbf{\Omega}^2$. We will subsequently refer to the structure discussed above as the *original structure*.

If the structure is subjected to an external harmonic excitation, $\mathbf{f} = \mathbf{f}_0 \sin \omega t$, where $\mathbf{f}_0 \in \mathbb{R}^n$ is a constant amplitude vector and ω is a constant driving frequency, then the equation of motion reads

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}_0 \sin \omega t. \tag{7}$$

A solution to this differential equation is given by

$$\mathbf{x} = \mathbf{F}(\omega)\mathbf{f}_0, \tag{8}$$

where

$$\mathbf{F}(\omega) = (-\omega^2\mathbf{M} + \mathbf{K})^{-1} = \sum_{i=1}^{n} \frac{\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}}{\omega_i^2 - \omega^2} \tag{9}$$

is the frequency response function of the structure, defined for $\omega^2 \neq \omega_i^2$.

One of the main objectives of structural design is to see to it that the driving frequency, ω, is well separated from the spectrum of the structure.

4. The constraint formulation

A set of *linear* constraints on the original structure is now introduced. These are defined by the following m, $1 \leqslant m < n$, independent linear equations:

$$\begin{cases} a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n = 0, \\ a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n = 0, \\ \quad\quad\quad \vdots \\ a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n = 0, \end{cases} \tag{10}$$

or, in more compact notation:

$$\mathbf{A}\mathbf{q} = \mathbf{0}_m, \tag{11}$$

where the constant matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$ is assumed to be of full rank, i.e. $\mathrm{rank}(\mathbf{A}) = m$. The configuration space of the constrained system is the linear subspace C in $\mathbb{R}^n$, defined by $C = \mathrm{kernel}(\mathbf{A}) = (\mathrm{range}(\mathbf{A}^{\mathrm{T}}))^{\perp}$ where $\dim(C) = k = n - m$.

Using the technique with Lagrangian multipliers it may be shown, cf. [9], that if $\mathbf{q} = \mathbf{q}(t)$ satisfies (1) and the constraint condition (10) then

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}_c\mathbf{q} = \mathbf{0}_n, \tag{12}$$

where

$$\mathbf{K}_c = \mathbf{K}_c(\mathbf{A}) = \mathbf{Q}(\mathbf{A})\mathbf{K} \tag{13}$$

is the so-called constrained stiffness matrix and

$$\mathbf{Q}(\mathbf{A}) = \mathbf{I}_{n\times n} - \mathbf{P}(\mathbf{A}), \quad \mathbf{P}(\mathbf{A}) = \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}(\mathbf{A})^{-1}\mathbf{A}\mathbf{M}^{-1}, \tag{14}$$

where the matrices $\mathbf{Q}$ and $\mathbf{P}$ represent projections, i.e. $\mathbf{Q}^2 = \mathbf{Q}$ and $\mathbf{P}^2 = \mathbf{P}$, and

$$\mathbf{\Gamma} = \mathbf{\Gamma}(\mathbf{A}) = \mathbf{A}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{m\times m} \tag{15}$$

is a symmetric and positive definite matrix. The equation

$$(-\omega^2\mathbf{M} + \mathbf{K}_c)\mathbf{x} = \mathbf{0}_n \tag{16}$$

represents the constrained eigenvalue problem.

Let the solution to the eigenvalue problem (16), i.e. the natural frequencies of the constrained structure and the corresponding mode shapes, be denoted by

$$\omega_{c,1}^2, \omega_{c,2}^2, \ldots, \omega_{c,n}^2 \quad \text{and} \quad \mathbf{x}_{c,1},\ \mathbf{x}_{c,2}, \ldots, \mathbf{x}_{c,n}. \tag{17}$$

The mode shapes of the constrained structure may be collected in the modal matrix $\mathbf{X}_c = [\mathbf{x}_{c,1}\ \ \mathbf{x}_{c,2} \ldots \mathbf{x}_{c,n}]$ and the corresponding natural frequencies in the spectral matrix $\mathbf{\Omega}_c^2 = \mathrm{diag}(\omega_{c,1}^2\ \ \omega_{c,2}^2 \ldots \omega_{c,n}^2)$. If $\mathbf{K}$ is positive semi-definite then the natural frequencies of the constrained structure, $\omega_{c,i}^2,\ \ 1 \leqslant i \leqslant n$, are real and nonnegative. The m first natural frequencies are equal to zero

$$\omega_{c,1}^2 = \omega_{c,2}^2 = \cdots = \omega_{c,m}^2 = 0 \tag{18}$$

and the following k natural frequencies are nonnegative

$$0 \leqslant \omega_{c,m+1}^2 \leqslant \omega_{c,m+2}^2 \leqslant \cdots \leqslant \omega_{c,n}^2. \tag{19}$$

The modal matrix $\mathbf{X}_c$ is nonsingular, and

$$\mathbf{X}_c^{\mathrm{T}} \mathbf{M} \mathbf{X}_c = \begin{bmatrix} \mathbf{G} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{H} & \mathbf{J} \end{bmatrix}, \tag{20}$$

where $\mathbf{G} \in \mathbb{R}^{m \times m}$ is symmetric, $\mathbf{H} \in \mathbb{R}^{k \times m}$, $k = n - m$ and

$$\mathbf{J} = \begin{bmatrix} \mathbf{0}_{s \times s} & \mathbf{0}_{s \times (s-r)} \\ \mathbf{0}_{(k-s) \times s} & \mathbf{I}_{(k-s) \times (k-s)} \end{bmatrix}, \tag{21}$$

where $0 \leqslant s \leqslant k$. Furthermore,

$$\mathbf{A}\mathbf{x}_{c,m+i} = \mathbf{0}_m, \quad i = 1, \ldots, k, \tag{22}$$

i.e. the mode shapes $\mathbf{x}_{c,m+1}, \mathbf{x}_{c,m+2}, \ldots, \mathbf{x}_{c,n}$ satisfy the constraint condition. For a proof of this result and a lengthy discussion on this subject see Ref. [9].

The first m modes, with natural frequencies all equal to zero, are *fictitious modes* for the constrained structure and should, from the point of view of physical interpretation, be ignored. Note that these modes do not satisfy the constraint condition, i.e. $\mathbf{A}\mathbf{x}_{c,i} \neq \mathbf{0}_m,\quad i = 1, \ldots, m$. The remaining nonnegative natural frequencies, according to Eq. (19), and their corresponding mode shapes

$$\mathbf{x}_{c,m+1}, \mathbf{x}_{c,m+2}, \ldots, \mathbf{x}_{c,n} \tag{23}$$

represent the vibration modes of the constrained structure.

Remark 1. The positive semi-definiteness of the stiffness matrix may, eventually, result in constrained *rigid body modes* with the spectrum

$$0 = \omega_{c,m+1}^2 = \omega_{c,m+2}^2 = \cdots = \omega_{c,m+s}^2 < \omega_{c,m+s+1}^2 \leqslant \cdots \leqslant \omega_{c,n}^2 \tag{24}$$

where the first s natural frequencies, $0 \leqslant s \leqslant k$, correspond to rigid body modes.

Remark 2. If the constraint matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ then the system would be fully constrained, i.e. it has no dofs and the generalized eigenvalue problem (16) reduces to

$$-\omega^2 \mathbf{M}\mathbf{x} = \mathbf{0}_n, \tag{25}$$

which of course, in accordance with Eq. (18), only has the trivial solutions, cf. Ref. [9].

5. The inverse structural modification problem

The spectrum mapping $\boldsymbol{\omega}_c^2 : \mathbb{R}^{m \times n} \to \mathbb{R}^n$of the eigenvalue problem (16) is given by

$$\boldsymbol{\omega}_c^2(\mathbf{A}) = [0 \quad 0 \quad \ldots \quad 0 \quad \omega_{c,m+1}^2(\mathbf{A}) \quad \omega_{c,m+2}^2(\mathbf{A}) \quad \ldots \quad \omega_{c,n}^2(\mathbf{A})]^{\mathrm{T}}. \tag{26}$$

From Eq. (16) it follows that

$$\boldsymbol{\omega}_{c,i}^2(\mathbf{A}) = \frac{\mathbf{x}_i(\mathbf{A})^{\mathrm{T}}\mathbf{K}_c(\mathbf{A})\mathbf{x}_i(\mathbf{A})}{\mathbf{x}_i(\mathbf{A})^{\mathrm{T}}\mathbf{M}\mathbf{x}_i(\mathbf{A})}. \tag{27}$$

Specifying the k natural frequencies of interest for the modified structure, with $1 \leqslant k \leqslant n$, results in a set of k nonlinear equations

$$\begin{cases} \omega_{c,m+1}^2(a_{ij}) = \omega_1^{2*}, \\ \omega_{c,m+2}^2(a_{ij}) = \omega_2^{2*}, \\ \qquad \vdots \\ \omega_{c,m+k}^2(a_{ij}) = \omega_k^{2*}, \end{cases} \tag{28}$$

where $\omega_i^{2*} \quad i = 1, \ldots, k$ denotes the desired natural frequencies.

A matrix $\mathbf{A}^* \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}(\mathbf{A}^*) = m$, satisfying (28), is said to be a solution to the structural modification problem. Consider the partial spectrum mapping $\tilde{\boldsymbol{\omega}}_c^2 : \mathbb{R}^{m\times n} \to \mathbb{R}^k$ defined by $\tilde{\omega}_{c,i}^2(\mathbf{A}) = \omega_{c,m+i}^2(\mathbf{A})$, $i = 1,\ldots,k$. The derivative of this mapping, the Jacobian, is denoted

$$\mathbf{J}(\mathbf{A}) = \frac{\partial \tilde{\omega}^2(\mathbf{A})}{\partial \mathbf{A}} \in \mathbb{R}^{k\times m\times n}. \tag{29}$$

Note that $\mathbf{J}(\mathbf{A})$ is a linear mapping from $\mathbb{R}^{m\times n}$ to $\mathbb{R}^k$. Since $k \leqslant p$ the solution of Eq. (28) will, in general, not be unique. For instance if $\mathrm{rank}(\mathbf{J}(\mathbf{A}^*)) \overset{\mathrm{def}}{=} \dim(\mathrm{range}(\mathbf{J}(\mathbf{A}^*))) = r$, $1 \leqslant r \leqslant k$, in a neighbourhood of $\mathbf{A}^*$ then the set of all solutions to Eq. (28) will be a manifold of dimension $p-r$ in a neighbourhood of $\mathbf{A}^*$. This manifold character of the solution set may eventually lead to a strong dependence of the numerical solution to Eq. (28), on the initial value for $\mathbf{A}$ taken in an iterative numerical solution procedure. We employ the notation $\tilde{\mathbf{x}}_i(\mathbf{A}) = \mathbf{x}_{m+i}(\mathbf{A})$.

Theorem. *The matrix elements J_{jl}^i of the Jacobian are given by*

$$\begin{aligned} J_{jl}^i &= \frac{\partial \omega_i^2}{\partial a_{jl}} \\ &= -2\tilde{\mathbf{x}}_i^{\mathrm{T}}\left(\mathbf{Q}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\boldsymbol{\Gamma}^{-1}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\frac{\partial \mathbf{A}}{\partial a_{jl}}\mathbf{Q}^{\mathrm{T}}\right)\mathbf{M}^{-1}\mathbf{K}\mathbf{Q}^{\mathrm{T}}\tilde{\mathbf{x}}_i, \end{aligned} \tag{30}$$

where $\tilde{\mathbf{x}}_i$ satisfy $\tilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{M}\tilde{\mathbf{x}}_i = 1$ and $i = 1,\ldots,k$, $j = 1,\ldots,m$, $l = 1,\ldots,n$.

Remark. Note that the matrix elements of the Jacobian do not contain derivatives of the eigenvectors.

For the proof of this Theorem we need the following Lemma:

Lemma. $\mathbf{x} \neq \mathbf{0}$, $\omega^2 > 0$ *is a solution to the eigenvalue problem* (16) *if and only if it is a solution to*

$$(-\omega^2\mathbf{I}_{n\times n} + \boldsymbol{\Psi})\boldsymbol{\eta} = \mathbf{0}, \tag{31}$$

where

$$\boldsymbol{\eta} = \mathbf{M}^{1/2}\mathbf{x} \tag{32}$$

and

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}(\mathbf{A}) = \boldsymbol{\Phi}(\mathbf{A})\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\boldsymbol{\Phi}(\mathbf{A}) \in \mathbb{R}^{n\times n} \tag{33}$$

is symmetric and positive semi-definite. Here

$$\boldsymbol{\Phi}(\mathbf{A}) = \mathbf{I}_{n\times n} - \boldsymbol{\Pi}(\mathbf{A}) \quad \textit{and} \quad \boldsymbol{\Pi}(\mathbf{A}) = \mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}(\mathbf{A})^{-1}\mathbf{A}\mathbf{M}^{-1/2} \tag{34}$$

are orthogonal projections, i.e. $\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{\Phi}$, $\boldsymbol{\Phi}^2 = \boldsymbol{\Phi}$ *and* $\boldsymbol{\Pi}^{\mathrm{T}} = \boldsymbol{\Pi}$, $\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}$.

Proof. A change of coordinates in Eq. (16), in accordance with Eq. (32), and a pre-multiplication of Eq. (16) with the matrix $\mathbf{M}^{-1/2}$ gives

$$(-\omega^2\mathbf{I}_{n\times n} + \mathbf{M}^{-1/2}\mathbf{K}_c\mathbf{M}^{-1/2})\boldsymbol{\eta} = \mathbf{0}, \tag{35}$$

which is equivalent to Eq. (16) and since

$$\begin{aligned}\mathbf{M}^{-1/2}\mathbf{K}_c\mathbf{M}^{-1/2} &= (\mathbf{I}_{n\times n} - \mathbf{M}^{-1/2}\mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1/2})\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\\ &= \boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2},\end{aligned} \tag{36}$$

Eq. (35) may be written as

$$(-\omega^2\mathbf{I}_{n\times n} + \boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2})\boldsymbol{\eta} = \mathbf{0}. \tag{37}$$

Now if $\omega^2 > 0$, by operating with $\boldsymbol{\Pi}$ on Eq. (37) gives $\boldsymbol{\Pi}\boldsymbol{\eta} = \mathbf{0}$ and consequently $\boldsymbol{\Phi}\boldsymbol{\eta} = \boldsymbol{\eta}$ and inserting this into Eq. (37) proves Eq. (31). Conversely, by operating with $\boldsymbol{\Pi}$ on Eq. (31) gives $\boldsymbol{\Pi}\boldsymbol{\eta} = \mathbf{0}$ and consequently $\boldsymbol{\Phi}\boldsymbol{\eta} = \boldsymbol{\eta}$ proving Eq. (35). This proves the Lemma. □

Remark. Note that $\boldsymbol{\Phi} = \mathbf{M}^{-1/2}\mathbf{Q}\mathbf{M}^{1/2}$ and $\boldsymbol{\Pi} = \mathbf{M}^{-1/2}\mathbf{P}\mathbf{M}^{1/2}$.

Proof of the Theorem. From the Lemma we obtain, for $i = 1,\ldots,k$

$$\tilde{\boldsymbol{\omega}}_i^2(\mathbf{A}) = \tilde{\boldsymbol{\eta}}_i(\mathbf{A})^{\mathrm{T}}\boldsymbol{\Psi}(\mathbf{A})\tilde{\boldsymbol{\eta}}_i(\mathbf{A}), \tag{38}$$

where $\tilde{\boldsymbol{\eta}}_i = \mathbf{M}^{1/2}\tilde{\mathbf{x}}_i$ and a direct calculation and using the symmetry of $\boldsymbol{\Psi}$ gives, cf. Ref. [22]

$$\begin{aligned}\frac{\partial\tilde{\omega}_i^2}{\partial a_{jl}} &= \frac{\partial\tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}}{\partial a_{jl}}\boldsymbol{\Psi}\tilde{\boldsymbol{\eta}}_i + \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}\frac{\partial\boldsymbol{\Psi}}{\partial a_{jl}}\tilde{\boldsymbol{\eta}}_i + \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}\boldsymbol{\Psi}\frac{\partial\tilde{\boldsymbol{\eta}}_i}{\partial a_{jl}}\\ &= \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}\frac{\partial\boldsymbol{\Psi}}{\partial a_{jl}}\tilde{\boldsymbol{\eta}}_i + 2\frac{\partial\tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}}{\partial a_{jl}}\boldsymbol{\Psi}\tilde{\boldsymbol{\eta}}_i\\ &= \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}\frac{\partial\boldsymbol{\Psi}}{\partial a_{jl}}\tilde{\boldsymbol{\eta}}_i + 2\tilde{\omega}_i^2\frac{\partial\tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}}{\partial a_{jl}}\tilde{\boldsymbol{\eta}}_i.\end{aligned} \tag{39}$$

Since $\tilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{M}\tilde{\mathbf{x}}_i = 1$, we get $\tilde{\boldsymbol{\eta}}_i(\mathbf{A})^{\mathrm{T}}\tilde{\boldsymbol{\eta}}_i(\mathbf{A}) = 1$ and thus

$$\frac{\partial\tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}}{\partial a_{jl}}\tilde{\boldsymbol{\eta}}_i = 0 \tag{40}$$

and this inserted into Eq. (39) gives

$$\begin{aligned}\frac{\partial\tilde{\omega}_i^2}{\partial a_{jl}} &= \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}\frac{\partial\boldsymbol{\Psi}}{\partial a_{jl}}\tilde{\boldsymbol{\eta}}_i\\ &= \tilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{M}^{1/2}\left(\frac{\partial\boldsymbol{\Phi}}{\partial a_{jl}}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\boldsymbol{\Phi} + \boldsymbol{\Phi}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\frac{\partial\boldsymbol{\Phi}}{\partial a_{jl}}\right)\mathbf{M}^{1/2}\tilde{\mathbf{x}}_i\\ &= 2\tilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{M}^{1/2}\frac{\partial\boldsymbol{\Phi}}{\partial a_{jl}}\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\boldsymbol{\Phi}\mathbf{M}^{1/2}\tilde{\mathbf{x}}_i.\end{aligned} \tag{41}$$

We then have to calculate the partial derivative

$$\frac{\partial\boldsymbol{\Phi}}{\partial a_{jl}} = -\frac{\partial\boldsymbol{\Pi}}{\partial a_{jl}} = -\mathbf{M}^{-1/2}\left(\frac{\partial\mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\boldsymbol{\Gamma}^{-1}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\frac{\partial\boldsymbol{\Gamma}^{-1}}{\partial a_{jl}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\frac{\partial\mathbf{A}}{\partial a_{j,l}}\right)\mathbf{M}^{-1/2} \tag{42}$$

and from $\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{-1} = \mathbf{I}_{m\times m}$ we get

$$\frac{\partial\boldsymbol{\Gamma}}{\partial a_{jl}}\boldsymbol{\Gamma}^{-1} + \boldsymbol{\Gamma}\frac{\partial\boldsymbol{\Gamma}^{-1}}{\partial a_{jl}} = \mathbf{0}_{m\times m} \tag{43}$$

and hence

$$\frac{\partial \mathbf{\Gamma}^{-1}}{\partial a_{jl}} = -\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}}{\partial a_{jl}}\mathbf{\Gamma}^{-1}. \tag{44}$$

By substituting Eq. (44) into Eq. (42) we get

$$\frac{\partial \mathbf{\Phi}}{\partial a_{jl}} = -\mathbf{M}^{-1/2}\left(\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\mathbf{\Gamma}^{-1}\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}}{\partial a_{jl}}\mathbf{\Gamma}^{-1}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{A}}{\partial a_{jl}}\right)\mathbf{M}^{-1/2} \tag{45}$$

and using Eq. (15)

$$\frac{\partial \mathbf{\Gamma}}{\partial a_{jl}} = \frac{\partial \mathbf{A}}{\partial a_{jl}}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{M}^{-1}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}, \tag{46}$$

which substituted into Eq. (45), leads to

$$\frac{\partial \mathbf{\Phi}}{\partial a_{jl}} = -\mathbf{M}^{-1/2}\left(\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\mathbf{\Gamma}^{-1}\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\left(\frac{\partial \mathbf{A}}{\partial a_{jl}}\mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{M}^{-1}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\right)\mathbf{\Gamma}^{-1}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{A}}{\partial a_{jl}}\right)\mathbf{M}^{-1/2}. \tag{47}$$

This can be rewritten as

$$\begin{aligned}\frac{\partial \mathbf{\Phi}}{\partial a_{jl}} = & -\mathbf{M}^{-1/2}((\mathbf{I}_{n\times n} - \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{A}\mathbf{M}^{-1})\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\mathbf{\Gamma}^{-1}\mathbf{A} \\ & + \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{A}}{\partial a_{jl}}(\mathbf{I}_{n\times n} - \mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{A}))\mathbf{M}^{-1/2} \\ = & -\mathbf{M}^{-1/2}\left(\mathbf{Q}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial a_{jl}}\mathbf{\Gamma}^{-1}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{A}}{\partial a_{jl}}\mathbf{Q}^{\mathrm{T}}\right)\mathbf{M}^{-1/2}.\end{aligned} \tag{48}$$

This, substituted into Eq. (41) and using the fact that $\mathbf{\Phi} = \mathbf{M}^{1/2}\mathbf{Q}^{\mathrm{T}}\mathbf{M}^{-1/2}$, proves the Theorem. □

Remark 1. It should be noted that for the general case, i.e. when no elements in the constraint matrix have been prescribed we get

$$\frac{\partial \mathbf{A}}{\partial a_{ij}} = \left[\frac{\partial a_{kl}}{\partial a_{ij}}\right] = [\delta_{ki}\delta_{lj}]. \tag{49}$$

Remark 2. If certain elements of the constraint matrix are prescribed to be zero, i.e. $a_{mn} = 0$ then of course we must let

$$\frac{\partial \mathbf{A}}{\partial a_{mn}} = \left[\frac{\partial a_{kl}}{\partial a_{mn}}\right] = [0]. \tag{50}$$

Depending on the desired spectral shift the number of constraints will vary from problem to problem. Using the Rayleigh separation theorem, cf. Ref. [26], in connection with the original spectrum, we can approximately predict the number of constraints needed. When the number of constraints m has been determined we can reduce the system by the same number of dofs and, hence, the m lowest eigenvalues will be equal to zero and, thus, the lowest eigenvalue to be regarded for the constrained structure is ω_{m+1}^2. In order for the problem to have a solution the number of constraint variables must be equal to or exceed the number of desired eigenvalues.

6. Numerical algorithm

A solution to Eq. (28) will be found by an iterative algorithm using Newton's method. Following Ref. [21], let $\mathbf{A}^i,\ i = 0, 1, \ldots.$ denote the value of the design variables at the ith iteration step. The next iterate is given by

$$\mathbf{A}^{i+1} = \mathbf{A}^i - \delta\mathbf{A}^i, \tag{51}$$

where the matrix $\delta\mathbf{A}^i$ is determined by solving the linear system

$$\mathbf{J}(\mathbf{A}^i)\delta\mathbf{A}^i = \tilde{\boldsymbol{\omega}}^2(\mathbf{A}^i) - \boldsymbol{\omega}^{2*}. \tag{52}$$

The iteration procedure is started by prescribing $\mathbf{A}^0$.

For practical computational purposes it is convenient to make the identification

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n} \leftrightarrow \mathbf{a} = \begin{bmatrix} a_{11} & \dots & a_{1n} & a_{21} & \dots & a_{2n}\dots & a_{m1} & \dots & a_{mn} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^p. \tag{53}$$

Then Eq. (52) may be written

$$\mathbf{J}_{\square}(\mathbf{a}^i)\delta\mathbf{a}^i = \tilde{\boldsymbol{\omega}}^2(\mathbf{a}^i) - \boldsymbol{\omega}^{2*}, \tag{54}$$

where $\mathbf{J}_{\square}$ is a rectangular matrix with k rows and p columns defined by $\mathbf{J}_{\square}(\mathbf{u})\mathbf{v} = \mathbf{J}(\mathbf{U})\mathbf{V}$.

Since $k<p$, Eq. (54) becomes an underdetermined system and in order to find a solution there are a few possibilities available. For instance, in Ref. [25], the Moore–Penrose pseudo-inverse is used. If rank$(\mathbf{J}_{\square}) = r \leqslant k$, a singular value decomposition

$$\mathbf{J}_{\square} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}} \tag{55}$$

is performed, where the matrices $\mathbf{U} \in \mathbb{R}^{k\times k}$ and $\mathbf{V} \in \mathbb{R}^{p\times p}$ are orthogonal matrices and $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_r) \in \mathbb{R}^{k\times p}$. Here $r = \min(k,p)$ and $\sigma_1 \geqslant \dots \geqslant \sigma_r \geqslant 0$, cf. Ref. [27]. By a partitioning of the matrix $\mathbf{V} \in \mathbb{R}^{p\times p}$, i.e. $\mathbf{V} = [\mathbf{V}_1 \quad \mathbf{V}_2], \mathbf{V}_1 \in \mathbb{R}^{p\times r}, \mathbf{V}_2 \in \mathbb{R}^{p\times(p-r)}$, a solution to Eq. (54) is found through

$$\delta\mathbf{a}^i = \mathbf{J}_{\square}^{+}(\mathbf{a}^i)\left(\boldsymbol{\omega}^2(\mathbf{a}^i) - \boldsymbol{\omega}^{2*}\right) + \mathbf{V}_2\mathbf{b} \tag{56}$$

where $\mathbf{J}_{\square}^{+}$ denotes the Moore–Penrose pseudo-inverse of $\mathbf{J}_{\square}$ and $\mathbf{b} \in \mathbb{R}^{p-r}$ is an arbitrary vector, see Ref. [25]. Note that Eq. (54) has an infinite number of solutions. For a detailed discussion of the Moore–Penrose pseudo-inverse, see Ref. [28], and for the calculations leading to Eq. (56), cf. [29].

The algorithm for solving Eq. (28) reads:

1. Calculate $\mathbf{M}^{-1}$ and determine the number of constraints m and the k eigenvalues of interest for the modified structure and put them in the vector $\boldsymbol{\omega}^{2*} \in \mathbb{R}^k$.
2. Set the iteration index $i = 0$. Choose an initial constraint matrix $\mathbf{A}^i$, and normalize the rows so that they have length equal to 1 ($\mathbf{A}^0$ can be randomly generated; however it must have full rank). Then calculate $\partial\mathbf{A}/\partial a_{ij}$ using Eq. (49).
3. Calculate $\boldsymbol{\Gamma}^i$ from Eq. (15) and its inverse $(\boldsymbol{\Gamma}^{-1})^i$ and then the projection $\mathbf{Q}^i$ using Eq. (14).
4. Solve the eigenvalue problem (16) and obtain the mass normalized mode shapes $\tilde{\mathbf{x}}_i(\mathbf{A}^i)$ and the eigenvalues $\tilde{\boldsymbol{\omega}}^{2i}$, break the loop when $\left\|\tilde{\boldsymbol{\omega}}^{2i} - \boldsymbol{\omega}^{2*}\right\|$ is sufficiently small.
5. Calculate the elements of the Jacobian $\mathbf{J}^i = \mathbf{J}(\mathbf{A}^i)$ using Eq. (30) and decompose into $\mathbf{J}_{\square}$.
6. Solve Eq. (54) for $\delta\mathbf{a}^i$. This can be done in a variety of ways, for instance by using the Moore–Penrose inverse of the Jacobian i.e. according to Eq. (56). Then decompose $\delta\mathbf{a}^i$ into $\delta\mathbf{A}^i$.
7. Update the vector $\mathbf{A}^i$ to $\mathbf{A}^{i+1}$ by using Eq. (51). Normalize the rows of $\mathbf{A}^{i+1}$ and repeat steps (3)–(7).

Remark. If we were to impose a constraint where some elements of the constraint matrix were prescribed to be zero, i.e. $a_{mn} = 0$ then we must set the initial value of $a_{mn}^0 = 0$ and instead of using Eq. (49) in step 2, use Eq. (50) when calculating $\partial\mathbf{A}/\partial a_{mn}$.

7. Examples

In all the examples below the solutions are generated taking $\mathbf{b}$ in Eq. (56) equal to the null vector.

Example 1. A free–free mass and spring structure consists of four springs and four masses that are coupled in series according to Fig. 1. All the springs have the stiffness $k_i = 10$ and all the masses are set equal to unity,

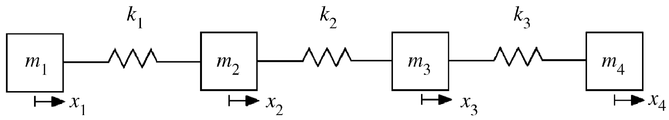

Fig. 1. The structure in Example 1.

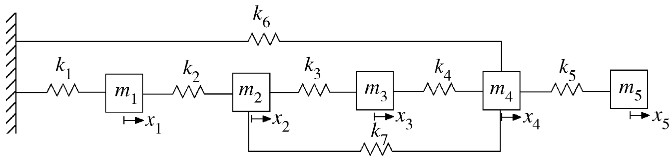

Fig. 2. The structure in Example 2.

i.e. $m_i = 1$. The original, unconstrained structure has the natural frequencies:

$$\boldsymbol{\omega}^2 = \begin{bmatrix} 0.0000 & 5.8579 & 20.0000 & 34.1421 \end{bmatrix}^{\mathrm{T}}.$$

The ambition is now to impose two constraints on the structure so that the lowest remaining two natural frequencies become

$$\boldsymbol{\omega}^{2*} = \begin{bmatrix} 15 & 25 \end{bmatrix}^{\mathrm{T}}.$$

By performing the iteration scheme described above, the constraint matrix

$$\mathbf{A} = \begin{bmatrix} 0.5973 & 0.0988 & 0.6981 & 0.3823 \\ -0.1025 & 0.7488 & 0.3637 & 0.5445 \end{bmatrix}.$$

is obtained. It should be noted that this solution is by no means unique, and that it depends on the initial constraint matrix $\mathbf{A}^0$ which may have been randomly generated.

Example 2. A mass and spring arrangement consisting of seven springs and five masses are coupled in series according to Fig. 2. The springs have the stiffnesses $k_i = 5i$ and the masses are set to $m_i = i$, so that the stiffness and mass matrices become

$$\mathbf{K} = \begin{bmatrix} 15 & -10 & 0 & 0 & 0 \\ -10 & 60 & -15 & -35 & 0 \\ 0 & -15 & 35 & -20 & 0 \\ 0 & -35 & -20 & 110 & -25 \\ 0 & 0 & 0 & -25 & 25 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

The original, unconstrained structure has the natural frequencies:

$$\boldsymbol{\omega}^2 = \begin{bmatrix} 1.8336 & 6.4916 & 13.5722 & 24.6768 & 42.5926 \end{bmatrix}^{\mathrm{T}}.$$

If the desired two lowest natural frequencies of the constrained structure are set to

$$\boldsymbol{\omega}^{2*} = \begin{bmatrix} 15 & 20 \end{bmatrix}^{\mathrm{T}}$$

it is obvious that in agreement with the Rayleigh separation theorem, cf. Ref. [26], we need to impose three constraints and a solution satisfying these requirements is

$$\mathbf{A} = \begin{bmatrix} 0.0408 & 0.0182 & 0.6769 & 0.7283 & 0.0970 \\ 0.5839 & 0.0156 & 0.0549 & 0.6842 & 0.4332 \\ 0.5517 & 0.1723 & 0.3333 & 0.6280 & 0.4006 \end{bmatrix}.$$

Example 3. If the desired two lowest natural frequencies of the same structure as in Example 2 are set to

$$\boldsymbol{\omega}^{2*} = \begin{bmatrix} 7 & 20 \end{bmatrix}^{\mathrm{T}}$$

then it is obvious that it is sufficient to use two constraints in order to meet this requirement. Thus the highest natural frequency is not specified and its numerical value will depend on the convergence direction, which in turn depends on the initial choice of the constraint matrix. This can be illustrated by randomly choosing five different initial constraint matrices:

$$\mathbf{A}_1^0 = \begin{bmatrix} 0.5238 & 0.4550 & 0.2779 & 0.3249 & 0.5796 \\ 0.1699 & 0.6752 & 0.4751 & 0.4546 & 0.2879 \end{bmatrix},$$

$$\mathbf{A}_2^0 = \begin{bmatrix} 0.3235 & 0.6550 & 0.3326 & 0.4424 & 0.3999 \\ 0.4591 & 0.2604 & 0.4166 & 0.5020 & 0.5440 \end{bmatrix},$$

$$\mathbf{A}_3^0 = \begin{bmatrix} 0.3411 & 0.4884 & 0.5623 & 0.3380 & 0.4634 \\ 0.1451 & 0.7669 & 0.5491 & 0.0894 & 0.2851 \end{bmatrix},$$

$$\mathbf{A}_4^0 = \begin{bmatrix} 0.0990 & 0.5819 & 0.7067 & 0.0416 & 0.3878 \\ 0.4113 & 0.4891 & 0.6979 & 0.2615 & 0.1900 \end{bmatrix},$$

$$\mathbf{A}_5^0 = \begin{bmatrix} 0.3734 & 0.3570 & 0.6016 & 0.4626 & 0.3965 \\ 0.0359 & 0.5100 & 0.5462 & 0.3143 & 0.5844 \end{bmatrix}.$$

Applying the algorithm to the problem and using the different initial constraint matrices yields the following solutions satisfying the requirements:

$$\mathbf{A}_1^5 = \begin{bmatrix} 0.4482 & 0.4204 & 0.2637 & 0.3272 & 0.6677 \\ 0.1761 & 0.6895 & 0.4635 & 0.4576 & 0.2633 \end{bmatrix},$$

$$\mathbf{A}_2^6 = \begin{bmatrix} 0.1570 & 0.6434 & 0.5023 & 0.4708 & 0.2958 \\ 0.4576 & 0.2323 & 0.2838 & 0.4699 & 0.6598 \end{bmatrix},$$

$$\mathbf{A}_3^4 = \begin{bmatrix} 0.3385 & 0.5018 & 0.5231 & 0.3341 & 0.4984 \\ 0.1283 & 0.7436 & 0.5986 & 0.0956 & 0.2512 \end{bmatrix},$$

$$\mathbf{A}_4^5 = \begin{bmatrix} -0.0016 & 0.5165 & 0.6592 & 0.0829 & 0.5401 \\ 0.6163 & 0.1525 & 0.7581 & -0.0905 & 0.1187 \end{bmatrix},$$

$$\mathbf{A}_5^4 = \begin{bmatrix} 0.3708 & 0.3271 & 0.6194 & 0.4506 & 0.4108 \\ 0.0384 & 0.5489 & 0.5213 & 0.3293 & 0.5631 \end{bmatrix}.$$

The corresponding natural frequencies are

$$\boldsymbol{\omega}_1^2 = \begin{bmatrix} 7 & 20 & 38.2630 \end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{\omega}_2^2 = \begin{bmatrix} 7 & 20 & 34.9009 \end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{\omega}_3^2 = \begin{bmatrix} 7 & 20 & 26.4797 \end{bmatrix}^{\mathrm{T}},$$

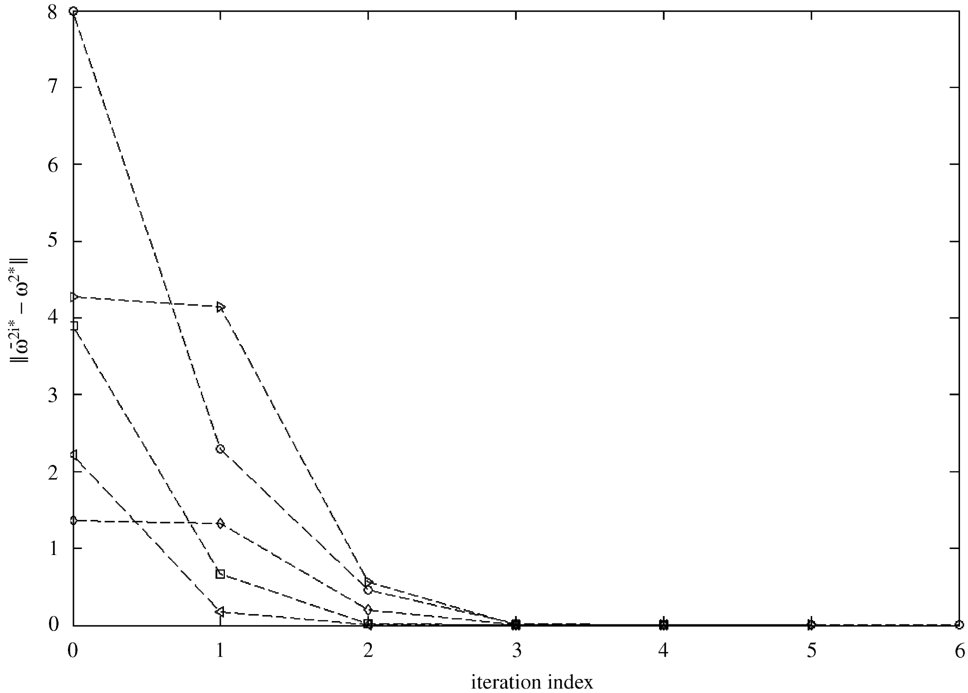

Fig. 3. The convergence for the different choices of initial constraint matrices in Example 3 (−−◇−−, $\mathbf{A}_1$; −−○−−, $\mathbf{A}_2$; −−□−−, $\mathbf{A}_3$; −−▷−−, $\mathbf{A}_4$ and −−◁−−, $\mathbf{A}_5$).

$$\boldsymbol{\omega}_4^2 = \begin{bmatrix} 7 & 20 & 33.3911 \end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{\omega}_5^2 = \begin{bmatrix} 7 & 20 & 32.5213 \end{bmatrix}^{\mathrm{T}}.$$

The convergence of $\left\|\tilde{\boldsymbol{\omega}}^{2i} - \boldsymbol{\omega}^{2*}\right\|$ for the different choices of initial constraint matrices in this example can be seen in Fig. 3, where obviously, the convergence rate is rather fast. It can also be seen that not only is the final solution highly dependent on the different choices of initial constraint matrices, but the number of iterations will also vary as a result of the different search directions which are initiated and is thus highly dependent on the starting point of the iteration scheme.

From a theoretical point of view, the constraints in these examples will always be physically realizable in the sense that they can always be reproduced, unlike methods which may generate negative cross-sectional areas or such, which of course never can be reproduced. From a practical point of view the constraints may be somewhat hard to reproduce since they do not necessarily have to be rigid interconnections or locking of dofs, however they may serve as guidance in order to obtain the desired spectrum.

8. Summary

In this paper an inverse structural modification problem, similar to the one described in Ref. [21], has been formulated. It uses the elements of a constraint matrix as design variables. The Jacobian for the reduced spectrum mapping was derived and, with a few minor modifications, it was used in an application of the solution scheme given in Ref. [25]. This solves the problem iteratively. The procedure is then applied numerically to a few simple problems in order to illustrate the methodology and convergence performance, as well as demonstrating the significance of the initial choice of the constraint matrix.

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